Kandala SS, Chakraborty S, Uchida TK, Vyasarayani CP. Hybrid method-of-receptances and optimization-based technique for pole placement in time-delayed systems. International Journal of Dynamics and Control 8(2): 558–569, 2020. doi: 10.1007/s40435-019-00570-5.

Hybrid method-of-receptances and optimization-based technique for pole placement in time-delayed systems

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Received: 14 May 2019 / Accepted: 10 August 2019

Abstract In this paper, we propose a pole-placement technique for second-order, time-delayed systems that combines the strengths of the method of receptances and an optimization-based strategy. The method of receptances involves solving an algebraic system of equations to obtain the closed-loop gains that place the poles of the system at desired locations. The method of receptances is simple and efficient, but the placed poles may not be the rightmost poles so the resulting closed-loop system may not be stable. By contrast, an optimization-based approach can explicitly consider the rightmost pole in the objective function and thus can guarantee its location. In this work, we use Galerkin approximations to obtain the characteristic roots of time-delayed systems. When the method of receptances provides an unsatisfactory solution, we use particle swarm optimization to improve the location of the rightmost pole. The proposed approach is demonstrated with numerical studies and is validated experimentally using a 3D hovercraft apparatus.

Keywords Time-delayed system · Pole placement · Stability · Method of receptances · Galerkin approximation · Particle swarm optimization

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1 Introduction

Consider the following second-order system with feedback control:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{b}u(t-\tau) \tag{1a}$$

$$u(t-\tau) = \mathbf{f}^{\mathrm{T}}\dot{\mathbf{x}}(t-\tau) + \mathbf{g}^{\mathrm{T}}\mathbf{x}(t-\tau) \tag{1b}$$

where $\mathbf{x}$ is the state vector; $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, damping, and stiffness matrices; u is the control effort, which is mapped onto the states by $\mathbf{b}$; $\mathbf{f}$ and $\mathbf{g}$ contain the control gains; and τ is a time delay. The system is governed by ordinary differential equations (ODEs) when $\tau = 0$ and delay differential equations (DDEs) when $\tau > 0$. In the latter case, the system is referred to as a time-delayed system (TDS). Time delays are inevitable in many practical systems, often appearing as a result of sensing, communication, and actuation processes.

Pole placement is a classical problem in the control theory domain. The objective is to design a controller that places the closed-loop poles at specific locations, thereby resulting in the desired system behaviour. In this work, we consider the pole-placement problem for systems governed by second-order DDEs. Many pole-placement techniques have been established for systems governed by ODEs, but pole placement for TDS remains an active area of investigation. Time delays in Eq. (1) turn a finite-dimensional system of ODEs into an infinite-dimensional system due to the transcendental nature of the characteristic equation. This infinite dimensionality makes the pole-placement problem challenging for TDS [10, 41, 42]. Because tuning an infinite number of parameters is impossible, our objective is to tune finitely many parameters to control an infinite-dimensional system.

The method of receptances (MoR) developed by Ram et al. [22, 23] is a popular algebraic strategy for addressing the pole-placement problem in TDS. Ram et al. [23] used MoR to place $2n$ poles (eigenvalues of the characteristic polynomial) of a second-order system with n degrees of freedom—that is, for a system with $2n$ states when written in first-order form. Later, Ram et al. [22] proposed a hybrid method for *partial* pole placement of second-order systems. As its name suggests, this work placed only $m < 2n$ poles at specified locations, leaving the remaining spectrum of $2n - m$ poles undisturbed. Although computationally straightforward, the MoR approach has several drawbacks: the poles placed at the specified locations may not be dominant (referred to as "spillover"), a separate analysis must be performed to determine whether the resulting closed-loop system is stable, and multiple delays cannot be accommodated. In Ram et al. [23], the time delay was handled using the Taylor series expansion; however, Insperger [9] demonstrated that the results obtained using Taylor series expansion for time delays are often inaccurate.

Several authors have advanced the field in recent years; here, we provide a brief chronology of these developments. Pratt et al. [20] defined the pole-placement problem for a TDS as a quadratic partial eigenvalue assignment problem with time delay, and proposed a "direct and partial modal" approach for active vibration problems. As with the method of Ram et al. [23], the approach proposed by Pratt et al. [20] was limited by its use of the Taylor series for incorporating the time delay and by the requirement to perform an *a posteriori* analysis to ascertain the stability of the resulting system. Ouyang and Singh [19] used MoR in one of the first applications of pole placement to asymmetric systems with time delay. Once again, this approach has the drawback that stability is not guaranteed.

Singh and Datta [30] obtained a closed-form solution to compute control gains for zero assignment in active vibration control problems. As with the work of Ram et al. [23], an *a posteriori* analysis is required to compute the primary eigenvalues of the system, which increases the complexity of computation. Bai et al. [5] formulated the pole-placement problem for second-order systems as a partial quadratic eigenvalue assignment problem (PQEAP) and proposed a multi-step hybrid method for solving symmetric systems. The proposed approach was applied to a multiple-input system, wherein the system matrices were combined with the measured receptances. A limitation of this approach is that it can be applied only to symmetric systems. Furthermore, the effects of high time delays (i.e., $\tau > 0.1$) were not explored. Bai et al. [4] later proposed an optimization-based approach to solve the PQEAP, extending the single-input hybrid method proposed by Ram et al. [22]. The optimization-based hybrid method of Bai et al. minimizes the feedback norms of the multi-input PQEAP with time delay.

Wang and Zhang [38] proposed a direct method to solve the partial eigenvalue assignment problem for high-order control systems with time delay, without first converting the system into first-order form. The proposed method requires only partial knowledge of the eigenvalues and corresponding eigenvectors of the matrix polynomial; however, the Taylor series was used to address the transcendental terms, which will fail to provide accurate results at higher delays. Mao and Dai [15] analyzed the sensitivity of closed-loop eigenvalues to perturbations in time delay during partial eigenvalue assignment. Li and Chu [13] generalized the well-known Kautsky, Nichols, and Van Dooren algorithm to solve the pole-placement problem for linear and quadratic TDS. They demonstrated that the results for systems with time delay are similar to those without delay, except for the presence of secondary eigenvalues.

Singh and Ouyang [32] proposed a method for assigning complex poles to second-order damped asymmetric systems using a constant-time-delay state-feedback controller. Again, an *a posteriori* analysis was necessary because the eigenvalues that are placed at the desired locations are not guaranteed to be the primary eigenvalues. Mao [14] proposed a partial eigenvalue assignment problem for TDS based on the orthogonality relations of the quadratic pencil (characteristic equation). Mao demonstrated the partial assignment of eigenvalues in a TDS without disturbing the remaining spectrum, obtained the explicit solution for the single-input case, and reported the parametric solution for the multi-input case. Singh et al. [31] defined a pole-placement problem without first transforming a given second-order system into a standard state-space form. Employing a sophisticated mathematical theory, Singh et al. were able to guarantee that the unassigned eigenvalues do not reside to the right of the assigned poles in the complex plane (i.e., there is no spillover).

Schmid and Nguyen [29] proposed a parametric formula for the feedback-gain matrix that will produce a desired set of closed-loop eigenvalues for a TDS. By considering only small time delays in the input, Schmid and Nguyen used unconstrained optimization to obtain the state-feedback matrix, minimizing the sensitivity of the eigenvalues to input delays. Schmid et al. [28] extended this approach to TDS with multiple time delays by first designing the control law for a non-delayed system, then investigating its applicability to the corresponding TDS. Schmid et al. demonstrated that it is possible to place the poles of a TDS at the same locations as for the system without delay. An *a posteriori* analysis using the quasi-polynomial root-finder (QPmR) algorithm [37] was performed to study the stability of the resulting system.

Zhang [43] proposed an explicit algorithm to assign the eigenvalues for multi-input, high-order control systems with time delay. Zhang demonstrated that this method avoids spillover and can be implemented with only partial information of the eigenvalues and corresponding eigenvectors of the matrix polynomial. Wang and Zhang extended their earlier work on partial assignment [38] and applied it to a multi-input TDS without use of the Sherman–Morrison formula [39]. Ariyatanapol et al. [3] proposed a receptance-based method for partial pole placement in asymmetric TDS that requires no knowledge of the mass, damping, or stiffness matrices. Ariyatanapol et al. used a single-input state-feedback controller and determined the critical stability of the system using the frequency-sweeping test. An *a posteriori* analysis was performed to calculate the first few dominant poles of the resulting closed-loop system and, thus, to analyze the stability of the system. Zhang and Shan [44] extended the work of Ram et al. [22] to solve the partial pole–zero placement problem in high-order systems using MoR.

Santos et al. [27] generalized the single-input, single-output, first-order small-gain theorem using the system receptances. Specifically, the small-gain theorem was extended to second-order systems with multiple inputs and time-varying delays with output feedback. Santos et al. also proposed a detuning strategy to address the tradeoff between performance and robustness with respect to variation in delay. Because the closed-loop poles are not computed in this method, their proposed approach can be used only to analyze delay uncertainty. Araújo [2] demonstrated use of system margins and Nyquist plots to determine the closed-loop stability of TDS. It was also shown that the Padé approximation for time delay in the frequency domain is as accurate as the corresponding truncated Taylor exponential expansion.

Experimental validation of stabilization techniques for TDS have not been widely explored in the literature. Previous studies have reported experimental validation for only a small number of control strategies, including semi-discretization [21], high-order control design [21], and cluster treatment of characteristic roots [18]. However, none of these studies have explored experimental validation of the pole-placement problem using real-time experiments.

In this work, we present a pole-placement technique for time-delayed systems that combines the strengths of the method of receptances and an optimization-based strategy. The method of receptances is simple and efficient, but may fail for certain systems and time delays. On the other hand, the optimization-based strategy guarantees the location of the rightmost pole but is more computationally demanding. Other established methods to design controllers for time-delayed systems include the Smith predictor, the modified Smith predictor, and finite spectrum assignment [17]; however, the performance of these methods depends on the accuracy of an internal model. For example, these techniques will stabilize an otherwise unstable time-delayed system only if the internal model is predicted accurately and if the effects of initial conditions and disturbances are known. These techniques are also sensitive to inaccuracies in the implementation of the control law, and require computing integrals of past control inputs which, when approximated using numerical quadrature, can affect system stability. By contrast, an internal model is not required in our proposed hybrid method, so it does not suffer from these limitations. Finally, we note that the Smith predictor, the modified Smith predictor, and finite spectrum assignment are sensitive to small perturbations in time delay around an assumed operating point; as we will show, our proposed approach exhibits robustness to perturbations in time delay.

We propose an optimization-based strategy to address the limitations of the method of receptances. The pole-placement method of Michiels et al. [16] also employs optimization; however, our approach differs in two substantial respects. First, Michiels et al. obtain the characteristic roots using subspace iteration [7] whereas we use Galerkin approximations. Second, Michiels et al. use a gradient descent algorithm for pole placement, which may fail to find the globally optimal solution; we propose particle swarm optimization to avoid converging to a local optimum.

The remainder of the paper is organized as follows. In Section 2, we briefly describe the method of receptances for completeness. In Section 3, we provide a detailed mathematical derivation of the Galerkin approximations we use to find the characteristic roots of quadratic TDS with a single delay. The optimization problem we solve is defined in Section 4. In Section 5, we apply the proposed method to stabilize examples given by Ram et al. [23]. Finally, we present experimental validation using a 3D hovercraft apparatus in Section 6.

2 Method of Receptances

We briefly describe the method of receptances (MoR) here for completeness. Consider the following system, which is obtained upon substituting $\mathbf{x}(t) = \mathbf{x}_0 e^{rt}$ into Eq. (1):

$$\left(r^2\mathbf{M} + r\mathbf{C} + \mathbf{K}\right)\mathbf{x}_0 e^{rt} = \left(r\mathbf{b}\mathbf{f}^{\mathrm{T}} + \mathbf{b}\mathbf{g}^{\mathrm{T}}\right)\mathbf{x}_0 e^{r(t-\tau)} \quad (2)$$

Eq. (2) can be rewritten as follows:

$$\left[r^2\mathbf{M} + r\left(\mathbf{C} - \mathbf{b}\mathbf{f}^{\mathrm{T}}e^{-r\tau}\right) + \left(\mathbf{K} - \mathbf{b}\mathbf{g}^{\mathrm{T}}e^{-r\tau}\right)\right]\mathbf{x}_0 = 0 \quad (3)$$

The receptance matrices associated with the open-loop system ($\mathbf{H}_o(r)$) and closed-loop system ($\mathbf{H}_c(r)$) are the following:

$$\mathbf{H}_o(r) = \left[r^2\mathbf{M} + r\mathbf{C} + \mathbf{K}\right]^{-1} \quad (4a)$$

$$\mathbf{H}_c(r) = \left[r^2\mathbf{M} + r\left(\mathbf{C} - \mathbf{b}\mathbf{f}^{\mathrm{T}}e^{-r\tau}\right) + \left(\mathbf{K} - \mathbf{b}\mathbf{g}^{\mathrm{T}}e^{-r\tau}\right)\right]^{-1} \quad (4b)$$

Matrix $\mathbf{H}_c(r)$ can be computed using the Sherman–Morrison formula [8]:

$$\mathbf{H}_c(r) = \mathbf{H}_o(r) + \frac{\mathbf{H}_o(r)\mathbf{b}\left(\mathbf{g} + r\mathbf{f}\right)^{\mathrm{T}}\mathbf{H}_o(r)e^{-r\tau}}{1 - \left(\mathbf{g} + r\mathbf{f}\right)^{\mathrm{T}}\mathbf{H}_o(r)\mathbf{b}e^{-r\tau}} \quad (5)$$

Note that the values of r that render $\mathbf{H}_c(r)$ unbounded are the eigenvalues of the closed-loop system. Thus, the characteristic equation of Eq. (5) is the following:

$$\left(\mathbf{g} + r\mathbf{f}\right)^{\mathrm{T}}\mathbf{H}_o(r)\mathbf{b} = e^{r\tau} \quad (6)$$

where the control vectors $\mathbf{f}$ and $\mathbf{g}$ can be computed given the system matrices ($\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$), the vector that maps the control effort onto the states ($\mathbf{b}$), the delay (τ), and eigenvalues r_k, $k = 1, 2, \ldots, 2n$:

$$\begin{bmatrix} r_1\mathbf{r}_1^{\mathrm{T}} & \mathbf{r}_1^{\mathrm{T}} \\ r_2\mathbf{r}_2^{\mathrm{T}} & \mathbf{r}_2^{\mathrm{T}} \\ \vdots & \vdots \\ r_{2n}\mathbf{r}_{2n}^{\mathrm{T}} & \mathbf{r}_{2n}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{f} \\ \mathbf{g} \end{Bmatrix} = \begin{Bmatrix} e^{r_1\tau} \\ e^{r_2\tau} \\ \vdots \\ e^{r_{2n}\tau} \end{Bmatrix} \quad (7)$$

where $\mathbf{r}_k \triangleq \mathbf{H}_o(r_k)\mathbf{b}$. Thus, the control vectors $\mathbf{f}$ and $\mathbf{g}$ can be obtained simply by solving a linear system of $2n$ equations (Eq. (7)) for $2n$ unknowns.

As stated in Section 1, the MoR approach is computationally straightforward but can suffer from spillover—that is, the poles placed at the specified locations may not be the dominant poles. A separate analysis must be performed to compute the characteristic roots and, thus, to determine whether the resulting closed-loop system is stable. We compute the roots of the TDS explicitly using Galerkin approximations with the spectral-tau method (described below). In situations where spillover is detected in the solution provided by the MoR approach, we propose a new optimization-based pole-placement strategy. The proposed optimization strategy makes use of the information about the characteristic roots that is provided by the Galerkin approximations.

A time-delayed system has a transcendental characteristic equation. Several methods have been proposed in the literature to compute the characteristic roots of a TDS, including the Lambert W function [42], Galerkin approximations [24, 35], semi-discretization [11], pseudospectral collocation [6], continuous-time approximation [33], and homotopy continuation [34]. In this work, Galerkin approximations are used to compute the characteristic roots. First, the equation governing the dynamics of the TDS (which is a DDE) is converted into a partial-differential equation (PDE) with boundary conditions. The PDE is then approximated by a system of ODEs, the eigenvalues of which are the approximate roots of the characteristic equation of the TDS. The efficacy of Galerkin approximations in studying the stability of DDEs has been demonstrated previously [24–26, 36]. These studies have also shown that the eigenvalues of the approximate ODE system converge to the eigenvalues of the original DDE system starting from the rightmost root.

The boundary conditions in previous work using Galerkin approximations have been handled using the spectral-tau and Lagrange multiplier methods [24–26, 36]. In this work, we use the spectral-tau method for embedding the boundary conditions because, with this method, the formulation can be generalized such that only the boundary conditions differ for different problems. Several options also exist for selecting the basis functions. In this work, we use shifted Legendre polynomials as the basis functions because of their superior convergence properties compared to other basis functions, such as mixed Fourier and Chebyshev polynomials [36].

3 Mathematical Modelling

In this section, we present the mathematical model for finding the characteristic roots of a DDE using Galerkin approximations. We consider systems of DDEs of the form given in Eq. (1) (higher-order systems have also been considered in the literature [12]). We begin by expressing Eq. (1) in first-order form:

$$\dot{\bar{\mathbf{x}}}(t) + \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{b}}u(t-\tau) = 0 \quad (8a)$$

$$u(t-\tau) = \bar{\mathbf{k}}^{\mathrm{T}}\bar{\mathbf{x}}(t-\tau) \quad (8b)$$

where $\bar{\mathbf{x}}(t) \triangleq \left[\dot{\mathbf{x}}^{\mathrm{T}}(t), \mathbf{x}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{P\times 1}$ is the state vector, u is the control effort, and $\tau > 0$ is the time delay. Matrix $\bar{\mathbf{A}} \in \mathbb{R}^{P\times P}$ and vectors $\bar{\mathbf{b}} \in \mathbb{R}^{P\times 1}$ and $\bar{\mathbf{k}} \in \mathbb{R}^{P\times 1}$ are given as follows:

$$\bar{\mathbf{A}} \triangleq \begin{bmatrix} \mathbf{M}^{-1}\mathbf{C} & \mathbf{M}^{-1}\mathbf{K} \\ -\mathbf{I} & 0 \end{bmatrix},$$
$$\bar{\mathbf{b}} \triangleq \begin{Bmatrix} -\mathbf{M}^{-1}\mathbf{b} \\ 0 \end{Bmatrix}, \qquad \bar{\mathbf{k}} \triangleq \begin{Bmatrix} \mathbf{f} \\ \mathbf{g} \end{Bmatrix} \tag{9}$$

where $\mathbf{I}$ is the identity matrix. The characteristic equation of Eq. (8) can be obtained by substituting $\bar{\mathbf{x}}(t) = \bar{\mathbf{x}}_0 e^{st}$ and equating the determinant to zero:

$$\det\left(s\mathbf{I} + \bar{\mathbf{A}} + \bar{\mathbf{b}}\bar{\mathbf{k}}^{\mathrm{T}} e^{-s\tau}\right) = 0 \tag{10}$$

Equation (10) is a quasi-polynomial due to the transcendental terms $e^{-s\tau}$ and therefore has infinitely many roots. These roots can be computed by formulating an abstract Cauchy problem, ultimately resulting in a large linear eigenvalue problem.

We first convert the system of DDEs (Eq. (8)) into a system of PDEs with time-dependent boundary conditions. We perform the following transformation:

$$\mathbf{y}(s,t) = \bar{\mathbf{x}}(t+s) \tag{11}$$

where $\mathbf{y}$ is a function of $s \in [-\tau, 0]$ and t. We obtain an abstract Cauchy problem by differentiating Eq. (11) with respect to s and t:

$$\frac{\partial \mathbf{y}(s,t)}{\partial t} = \frac{\partial \bar{\mathbf{x}}(t+s)}{\partial (t+s)} \frac{\partial (t+s)}{\partial t} = \frac{\partial \bar{\mathbf{x}}(t+s)}{\partial (t+s)} \tag{12a}$$
$$\frac{\partial \mathbf{y}(s,t)}{\partial s} = \frac{\partial \bar{\mathbf{x}}(t+s)}{\partial (t+s)} \frac{\partial (t+s)}{\partial s} = \frac{\partial \bar{\mathbf{x}}(t+s)}{\partial (t+s)} \tag{12b}$$

Equating Eqs. (12a) and (12b) results in the following PDE:

$$\frac{\partial \mathbf{y}(s,t)}{\partial t} = \frac{\partial \mathbf{y}(s,t)}{\partial s}, \quad s \in [-\tau, 0] \tag{13}$$

The boundary conditions for Eq. (13) can be computed from Eq. (11) upon substituting $s = 0$ and $s = -\tau$:

$$\mathbf{y}(0,t) = \bar{\mathbf{x}}(t) \tag{14a}$$
$$\mathbf{y}(-\tau,t) = \bar{\mathbf{x}}(t-\tau) \tag{14b}$$

Differentiating Eq. (14a) with respect to t provides the following relationship between $\mathbf{y}(s,t)$ and the state derivatives $\dot{\bar{\mathbf{x}}}(t)$:

$$\left.\frac{\partial \mathbf{y}(s,t)}{\partial t}\right|_{s=0} = \dot{\bar{\mathbf{x}}}(t) \tag{15}$$

Finally, we combine Eq. (14) with Eq. (8):

$$\left.\frac{\partial \mathbf{y}(s,t)}{\partial t}\right|_{s=0} + \bar{\mathbf{A}}\mathbf{y}(0,t) + \bar{\mathbf{b}}\bar{\mathbf{k}}^{\mathrm{T}}\mathbf{y}(-\tau,t) = 0 \tag{16}$$

We now approximate the solution of the PDE given in Eq. (13) with the following series:

$$y_i(s,t) = \sum_{j=1}^{\infty} \phi_{ij}(s)\eta_{ij}(t), \quad i = 1, 2, \ldots, P \tag{17}$$

where $\phi_{ij}(s)$ are the basis functions, $\eta_{ij}(t)$ are the coordinates (which are time dependent), i is the index into the state vector $\bar{\mathbf{x}}(t)$, and j is the corresponding term in each basis function. In this work, we use shifted Lagrange polynomials as the basis functions:

$$\phi_1(s) = 1 \tag{18a}$$
$$\phi_2(s) = 1 + \frac{2s}{\tau} \tag{18b}$$
$$\phi_k(s) = \frac{(2k-3)\,\phi_2(s)\phi_{k-1}(s) - (k-2)\,\phi_{k-2}(s)}{k-1}, \quad k = 3, 4, \ldots, N \tag{18c}$$

Shifted Lagrange polynomials are selected for their superior convergence properties, as shown in previous studies (e.g., [36]). We truncate the infinite series (Eq. (17)) at N terms:

$$y_i(s,t) \approx \boldsymbol{\phi}_i^{\mathrm{T}}(s)\boldsymbol{\eta}_i(t), \quad i = 1, 2, \ldots, P \tag{19}$$

where $\boldsymbol{\phi}_i(s) \triangleq [\phi_{i1}(s), \phi_{i2}(s), \ldots, \phi_{iN}(s)]^{\mathrm{T}}$ and $\boldsymbol{\eta}_i(t) \triangleq [\eta_{i1}(t), \eta_{i2}(t), \ldots, \eta_{iN}(t)]^{\mathrm{T}}$. For simplicity of notation, we define the following:

$$\boldsymbol{\Phi}(s) \triangleq \begin{bmatrix} \boldsymbol{\phi}_1(s) & 0 & \cdots & 0 \\ 0 & \boldsymbol{\phi}_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\phi}_P(s) \end{bmatrix} \in \mathbb{R}^{NP\times P} \tag{20a}$$
$$\boldsymbol{\beta}(t) \triangleq \left[\boldsymbol{\eta}_1^{\mathrm{T}}(t), \boldsymbol{\eta}_2^{\mathrm{T}}(t), \ldots, \boldsymbol{\eta}_P^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{NP\times 1} \tag{20b}$$

and express Eq. (19) in vector form:

$$\mathbf{y}(s,t) = \left[\boldsymbol{\phi}_1^{\mathrm{T}}(s)\boldsymbol{\eta}_1(t), \boldsymbol{\phi}_2^{\mathrm{T}}(s)\boldsymbol{\eta}_2(t), \ldots, \boldsymbol{\phi}_P^{\mathrm{T}}(s)\boldsymbol{\eta}_P(t)\right]^{\mathrm{T}} = \boldsymbol{\Phi}^{\mathrm{T}}(s)\boldsymbol{\beta}(t) \tag{21}$$

We now obtain a system of ODEs by substituting the series solution (Eq. (21)) into the PDE (Eq. (13)), premultiplying the result by $\boldsymbol{\Phi}(s)$, and integrating over the domain $s \in [-\tau, 0]$:

$$\mathbf{G}\ddot{\boldsymbol{\beta}}(t) = \mathbf{H}\dot{\boldsymbol{\beta}}(t) \tag{22}$$

where $\mathbf{G}$ and $\mathbf{H}$ are square, block-diagonal matrices of dimension NP:

$$\mathbf{G} \triangleq \begin{bmatrix} \mathbf{G}^{(1)} & 0 & \cdots & 0 \\ 0 & \mathbf{G}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{G}^{(P)} \end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{H} \triangleq \begin{bmatrix} \mathbf{H}^{(1)} & 0 & \cdots & 0 \\ 0 & \mathbf{H}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H}^{(P)} \end{bmatrix}^{\mathrm{T}} \tag{23}$$

Submatrices $\mathbf{G}^{(i)}$ and $\mathbf{H}^{(i)}$ are defined as follows:

$$\mathbf{G}^{(i)} \triangleq \int_{-\tau}^{0} \phi_i(s)\phi_i^{\mathrm{T}}(s)\mathrm{d}s, \quad \mathbf{H}^{(i)} \triangleq \int_{-\tau}^{0} \phi_i(s)\phi_i'(s)^{\mathrm{T}}\mathrm{d}s \tag{24}$$

where $\phi_i'(s)$ denotes the derivative of $\phi_i(s)$ with respect to s. Note that the use of shifted Lagrange polynomials as basis functions allows us to express submatrices $\mathbf{G}^{(i)}$ and $\mathbf{H}^{(i)}$ in closed form [36].

The boundary conditions that transform the initial value problem into an initial–boundary value problem are obtained by substituting Eq. (11) into Eq. (8), where $\mathbf{y}(s,t)$ is given by Eq. (21):

$$\mathbf{\Phi}^{\mathrm{T}}(0)\dot{\boldsymbol{\beta}}(t) + \left[\bar{\mathbf{A}}\mathbf{\Phi}^{\mathrm{T}}(0) + \bar{\mathbf{b}}\bar{\mathbf{k}}^{\mathrm{T}}\mathbf{\Phi}^{\mathrm{T}}(-\tau)\right]\boldsymbol{\beta}(t) = 0 \tag{25}$$

We embed the boundary conditions (Eq. (25)) into the ODE system (Eq. (22)) using the spectral-tau method:

$$\tilde{\mathbf{G}}\ddot{\boldsymbol{\beta}}(t) = \tilde{\mathbf{H}}\dot{\boldsymbol{\beta}}(t) + \tilde{\mathbf{K}}\boldsymbol{\beta}(t) \tag{26}$$

where $\tilde{\mathbf{G}}$, $\tilde{\mathbf{H}}$, and $\tilde{\mathbf{K}}$ are the matrices obtained upon replacing every iN-th row of Eq. (22) with the i-th row of Eq. (25) for $i = 1, 2, \ldots, P$. Finally, we define state vector $\boldsymbol{\zeta} \triangleq \left[\dot{\boldsymbol{\beta}}(t)^{\mathrm{T}}, \boldsymbol{\beta}(t)^{\mathrm{T}}\right]^{\mathrm{T}}$ and rewrite Eq. (26) as follows:

$$\dot{\boldsymbol{\zeta}}(t) = \mathbf{Z}\boldsymbol{\zeta}(t) \tag{27}$$

where the eigenvalues of $\mathbf{Z}$ can be used to study the stability of the system.

Equation (27) is a system of ODEs whose response approximates that of the original system of DDEs (Eq. (8)). As N (the number of terms retained in the series solution, Eq. (19)) increases, the eigenvalues of $\mathbf{Z}$ converge to the characteristic roots of Eq. (8) [36]. We define the absolute error ϵ as the value of the characteristic equation (Eq. (10)) upon substitution of the eigenvalues of $\mathbf{Z}$. In this work, we define the convergence criterion to be $\epsilon < 10^{-4}$ and, thus, obtain the spectrum of the original DDE system (Eq. (8)) from the ODE system (Eq. (27)).

4 Problem Definition

Consider the second-order system given by Eq. (1). Given $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, and τ, we wish to determine the feedback gains $\mathbf{f}$ and $\mathbf{g}$ that place all roots in the left half of the complex plane and create a specified spectral gap between the rightmost root and the imaginary axis. We find the gains $\mathbf{f}$ and $\mathbf{g}$ by minimizing the following objective function:

$$J = \left(\mathrm{Re}\left\{\lambda_{\max}(\mathbf{f}, \mathbf{g})\right\} + \alpha\right)^2 \tag{28}$$

where $\mathrm{Re}\{\lambda_{\max}\}$ is the real part of the rightmost eigenvalue, which is a function of feedback gains $\mathbf{f}$ and $\mathbf{g}$, and $\alpha > 0$ is the desired spectral gap. If a solution $(\mathbf{f}^*, \mathbf{g}^*)$ is obtained where $J(\mathbf{f}^*, \mathbf{g}^*) = 0$, then the rightmost root is placed at the desired location—that is, $\mathrm{Re}\{\lambda_{\max}\} = -\alpha$. However, if $J(\mathbf{f}^*, \mathbf{g}^*) > 0$, then the rightmost root is not placed at the desired location (i.e., the spectral gap is less than α), but the precise location of the rightmost root is still obtained and, thus, the stability of the system can be determined. In practice, it may be necessary to accompany the objective function (Eq. (28)) with constraints on the feedback gains $\mathbf{f}$ and $\mathbf{g}$. In Section 6, we solve a constrained optimization problem whose constraints ensure the gains are within a physically realizable range. In this work, the objective function given by Eq. (28) is minimized using the particle swarm optimization (PSO) technique. PSO is a widely used swarm-intelligence-based algorithm due to its simplicity, flexibility, and ease of implementation [40].

5 Results and Discussion

In the domain of algebraic frameworks, the method of receptances (MoR) has gained popularity because it provides analytical expressions to solve the pole-placement problem in systems governed by DDEs. In this section, we demonstrate the strengths and limitations of the MoR approach, and we employ Galerkin approximations to obtain the characteristic roots corresponding to the solutions obtained using MoR. When the MoR approach does not achieve the desired spectral gap, we use the proposed optimization-based technique to improve the solution.

5.1 Example 1

Consider the system obtained upon substituting the following matrices into Eq. (1) (from [23]):

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{29}$$

We use MoR to place the eigenvalues of this system at two location sets: $S_1 \triangleq [-1, -1 \pm i, -2]$ (from [23]) and $S_2 \triangleq [-0.5, -1 \pm i, -2]$. It was observed that, for some delays $\tau \in [0.001, 1.5]$, the system becomes unstable when the desired location of the rightmost roots is S_1. Using location set S_2, spillover of the dominant roots still occurs but the system remains stable for the entire range of τ. Galerkin approximations are used to obtain the characteristic roots for this system, where feedback gains $\mathbf{f}$ and $\mathbf{g}$ are obtained using MoR. Figure 1 illustrates the locations of the three rightmost roots as delay τ varies, using target pole locations S_1 and S_2. As shown, spillover is evident in both cases—that is, for some values of delay τ, the real part of the rightmost root is not in the desired location. Specifically, for location set S_1 (Fig. 1(a)), there is always a pole at -1 as desired but, for some delays τ, there is also a pole to the right of -1; in the case of location set S_2 (Fig. 1(b)), there is a pole at -0.5 as desired but there is also a pole to the right of –0.5 for some values of delay τ. It is important to note that, in some situations, spillover of the dominant roots will result in an unstable closed-loop system.

We now solve the pole-placement problem for the same system using the proposed optimization-based strategy. Figure 2 illustrates the location of the rightmost root as delay τ varies, using $\alpha = 1$ and $\alpha = 0.5$ in the objective function (Eq. (28)). These values of α correspond to the spectral gaps described by location sets S_1 and S_2. When $\alpha = 0.5$ (Fig. 2(b)), $\mathrm{Re}\{\lambda_{\max}\} \approx -0.5$ for all delays τ—a substantial improvement over the performance of the MoR approach. An improvement is also observed when $\alpha = 1$ (Fig. 2(a)): although spillover still occurs over a similar range of τ, the deviation from the desired pole location is reduced significantly. Similar results are observed when using different optimization parameters, such as the relative tolerance and the number of generations. Because Galerkin approximations are used in the proposed optimization-based strategy, the stability of the system can be evaluated in cases of spillover without requiring any additional analysis.

5.2 Example 2

We now consider a second-order example given in Ram et al. [23], where $\mathbf{M} = 1$, $\mathbf{C} = 0.01$, $\mathbf{K} = 5$, and $\mathbf{b} = 1$ in Eq. (1):

$$\ddot{x}(t) + 0.01\dot{x}(t) + 5x(t) = f\dot{x}(t-\tau) + gx(t-\tau) \qquad (30)$$

where delay $\tau > 0$. We again compare the performance of the MoR approach with that of the optimization-based strategy, using a location set of $S \triangleq [-0.5, -47]$ for the former and $\alpha = 0.5$ for the latter. Figure 3 illustrates the locations of the two rightmost roots of Eq. (30) as delay τ varies, with feedback gains $\mathbf{f}$ and $\mathbf{g}$ determined using the MoR approach and the optimization-based strategy. We observe spillover using the MoR approach (Fig. 3(a)) when $\tau \in [0.093, 0.210]$ and $\tau > 0.838$, resulting in instability for delays exceeding 1.134 seconds. By contrast, there is no spillover of the rightmost root for any value of τ and the system is never unstable when the optimization-based strategy is used (Fig. 3(b)).

6 Experimental Validation

We validated the proposed approach experimentally using a 3D hovercraft apparatus (Quanser Inc., Markham, Ontario, Canada), as shown in Fig. 4. The hovercraft is a decoupled system—that is, its motion about the yaw, pitch, and roll axes is decoupled. Our experiments comprised motion about only the yaw axis, which nevertheless required coordination of all four motors. The experimental apparatus has an inherent time delay of 2 ms, which is well below the critical delays encountered in this study. The equation governing the motion of this system about the yaw axis is given as follows [1]:

$$\ddot{\theta}_y = -0.1304\big(f\dot{\theta}_y(t-\tau) + g\theta_y(t-\tau)\big) \qquad (31)$$

where θ_y is the yaw angle. As shown in Fig. 5(a), the feedback gains f and g computed using the MoR approach result in an unstable system for delays exceeding $\tau = 131$ ms. The optimization-based strategy (with $\alpha = 6$) increased the amount of delay that can be tolerated to $\tau = 194$ ms (Fig. 5(b)).

To validate these results, we deliberately introduced a delay into the experimental system and computed feedback gains using the proposed optimization-based strategy for four values of τ: 131 ms, 140 ms, 150 ms, and 160 ms (Table 1). The system response for a delay of $\tau = 131$ ms is shown in Fig. 6(a). A square waveform input of magnitude $\pm 5°$ was provided as the reference trajectory. Clearly, the feedback gains obtained using the optimization-based strategy resulted in a stable system; gains obtained using the MoR approach result in instability. Figures 6(b) and 7 illustrate the system response for the same reference signal when the delay is increased beyond 131 ms. As shown, the system response remains stable in all cases. The motor voltages for a delay of 160 ms (corresponding to the yaw angle shown in Fig. 7(b)) are shown in Fig. 8.

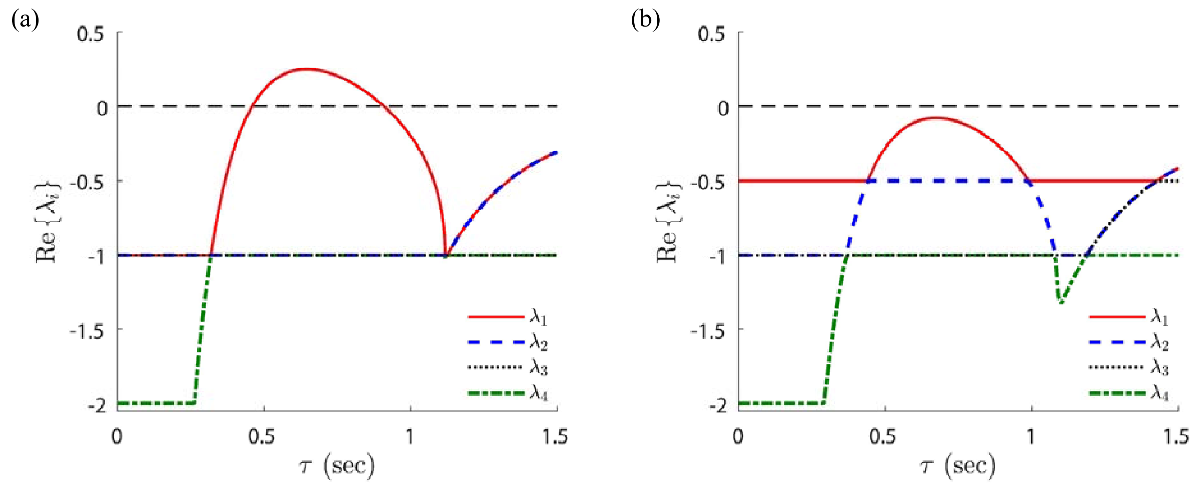

Fig. 1 Locations of the four roots of Eq. (1) as delay τ varies, using MoR with location sets (a) S_1 and (b) S_2. Matrices **M**, **C**, and **K** are given by Eq. (29).

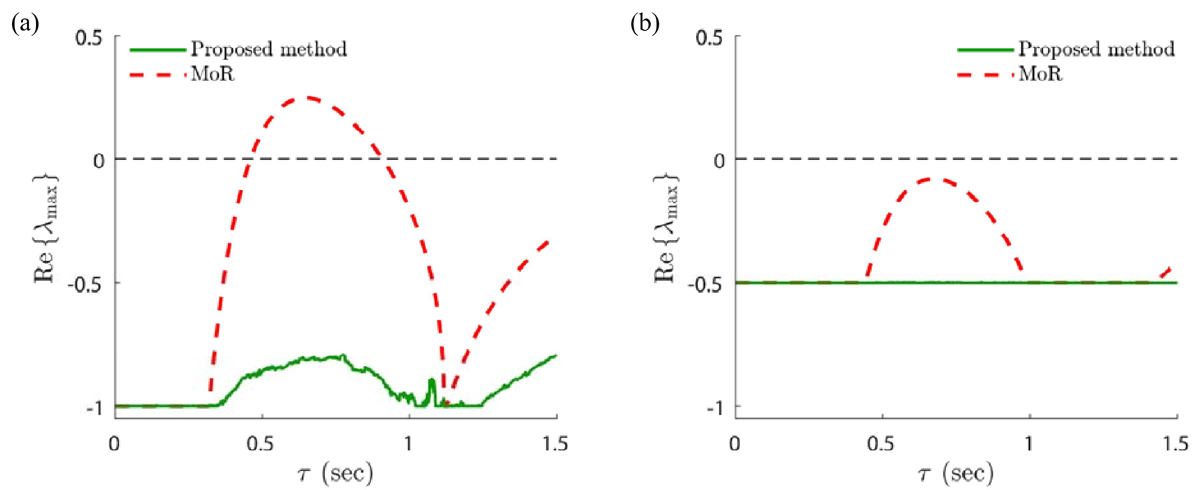

Fig. 2 Location of the rightmost root of Eq. (1) as delay τ varies, using the proposed optimization-based strategy with (a) $\alpha = 1$ and (b) $\alpha = 0.5$. Matrices **M**, **C**, and **K** are given by Eq. (29). The rightmost roots from Fig. 1 are shown for comparison (dashed lines).

Table 1 Feedback gains obtained using the proposed optimization-based strategy for the 3D hovercraft apparatus.

	Feedback gains	
Delay τ (ms)	f	g
131	44.2624	111.8034
140	43.2896	111.8034
150	42.2095	111.8034
160	41.1300	111.8034

7 Conclusions and Future Work

In this work, a hybrid method-of-receptances and optimization-based technique has been proposed to solve the pole-placement problem in time-delayed systems. Using examples from the literature, it has been demonstrated that the MoR approach can place the dominant root to the right of the specified location, resulting in a deficient spectral gap and potentially an unstable closed-loop system. An optimization-based strategy is proposed to complement the MoR approach by providing improved feedback gains for those delays where the MoR solution is unacceptable. The efficacy of this strategy was demonstrated using examples from the literature. Experimental validation was performed using a 3D hovercraft apparatus with a deliberately introduced delay. We demonstrated that the optimization-based strategy was able to stabilize the hovercraft for delays exceeding those that can be accommodated by the MoR approach.

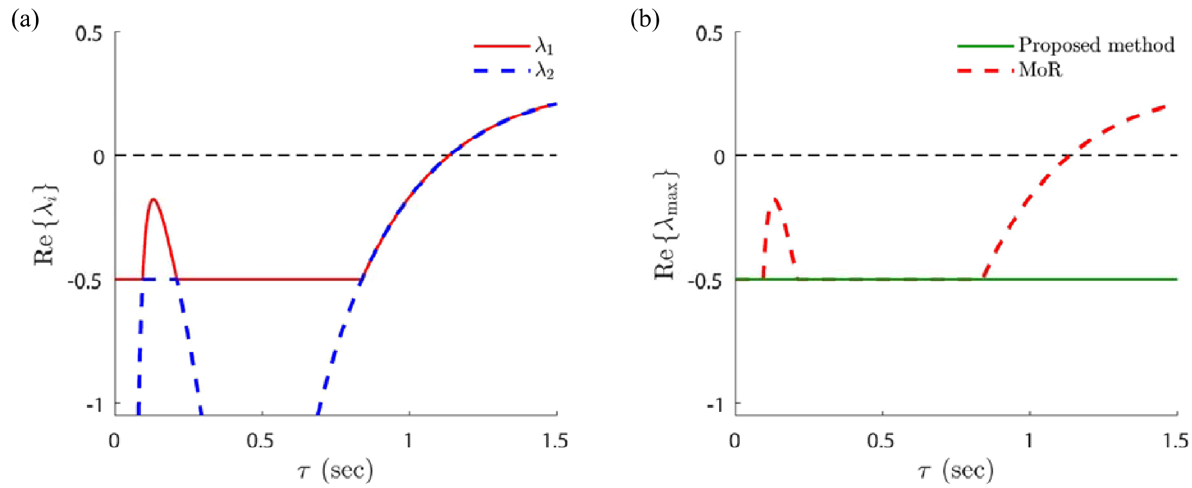

Fig. 3 Locations of the rightmost roots of Eq. (30) as delay τ varies: (a) the two rightmost roots obtained using the MoR approach, and (b) the rightmost root obtained using the proposed optimization-based strategy with $\alpha = 0.5$. The rightmost root from panel (a) is displayed in panel (b) for comparison (dashed line).

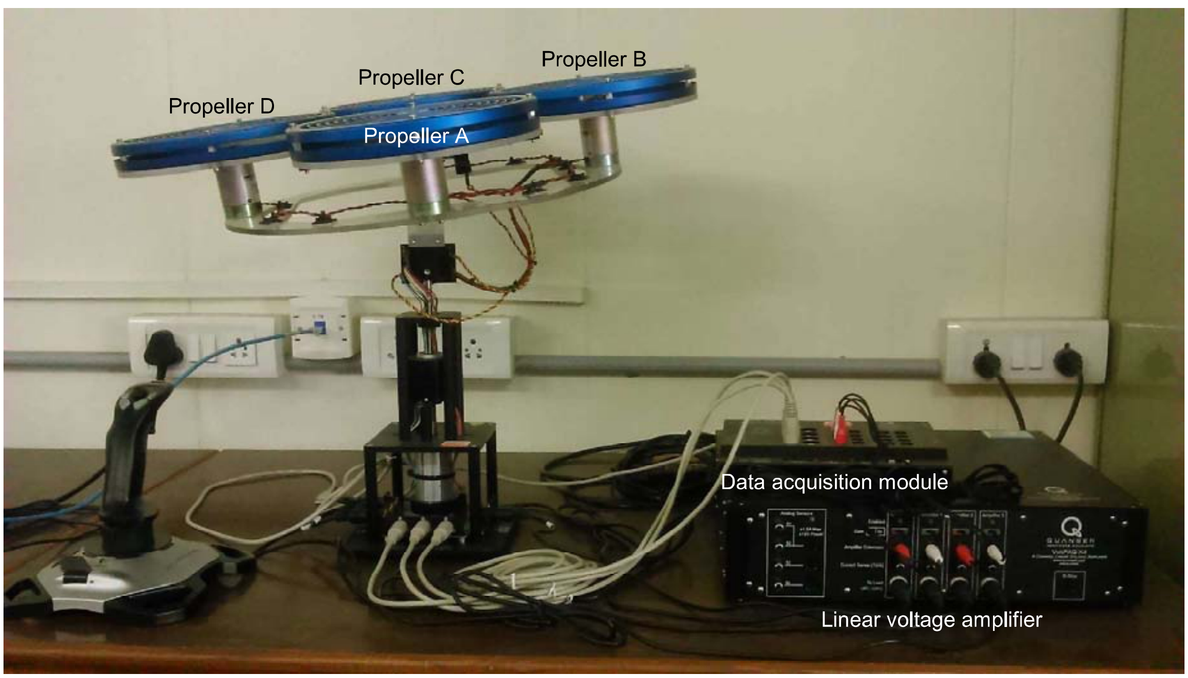

Fig. 4 3D hovercraft apparatus used for experimental validation.

Thus, the proposed hybrid method-of-receptances and optimization-based technique expands the range of time-delayed systems to which pole placement can be applied.

Directions for future work include extending the method of receptances to handle higher-order systems, time-varying delays, and constraints imposed by system parameters, actuator saturation, and limits on feedback gains. These constraints can be readily incorporated into the optimization-based technique to study in detail their effects on system stability. The effects of delays in the output equations will also be investigated.

Acknowledgements

We thank Dr. Ketan Detroja for providing laboratory space for the 3D hovercraft apparatus used in Section 6, and K. Subhash for his assistance performing the experiments.

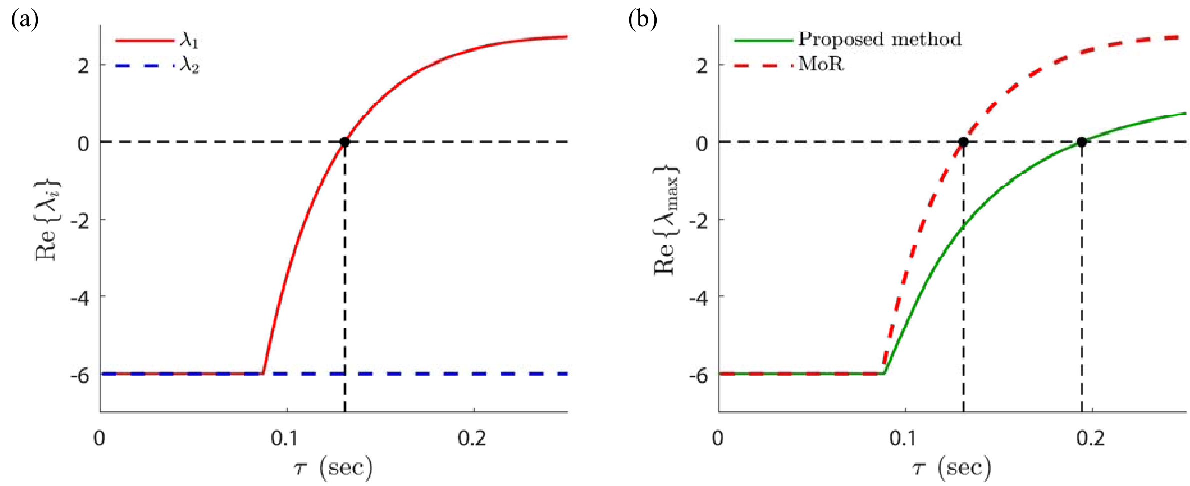

Fig. 5 Locations of the rightmost roots of Eq. (31) as delay τ varies: (a) the two rightmost roots obtained using the MoR approach, and (b) the rightmost root obtained using the proposed optimization-based strategy with $\alpha = 6$. The rightmost root from panel (a) is displayed in panel (b) for comparison (dashed line).

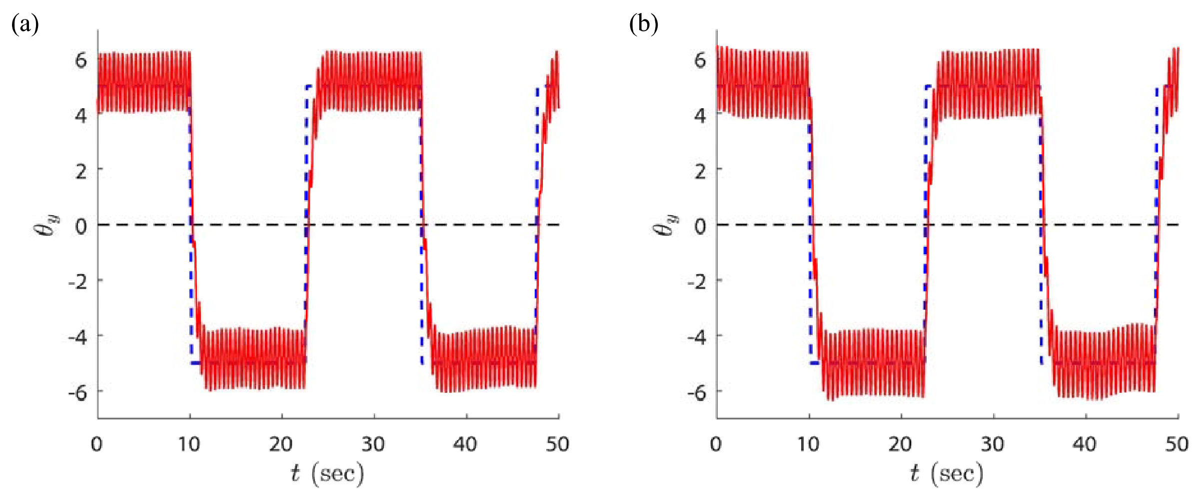

Fig. 6 Yaw angle ($\theta_y(t)$) of 3D hovercraft apparatus with feedback gains obtained using the proposed optimization-based strategy and delays of (a) 131 ms and (b) 140 ms. The reference signal is also shown (dashed line).

Competing Interests

The authors have no competing interests to declare.

Funding Statement

Funding was provided to C.P. Vyasarayani by the Department of Technology through the Inspire fellowship (Grant Number DST/INSPIRE/04/2014/000972). The funders had no role in study design, data collection and analysis, decision to publish, or preparation of the manuscript.

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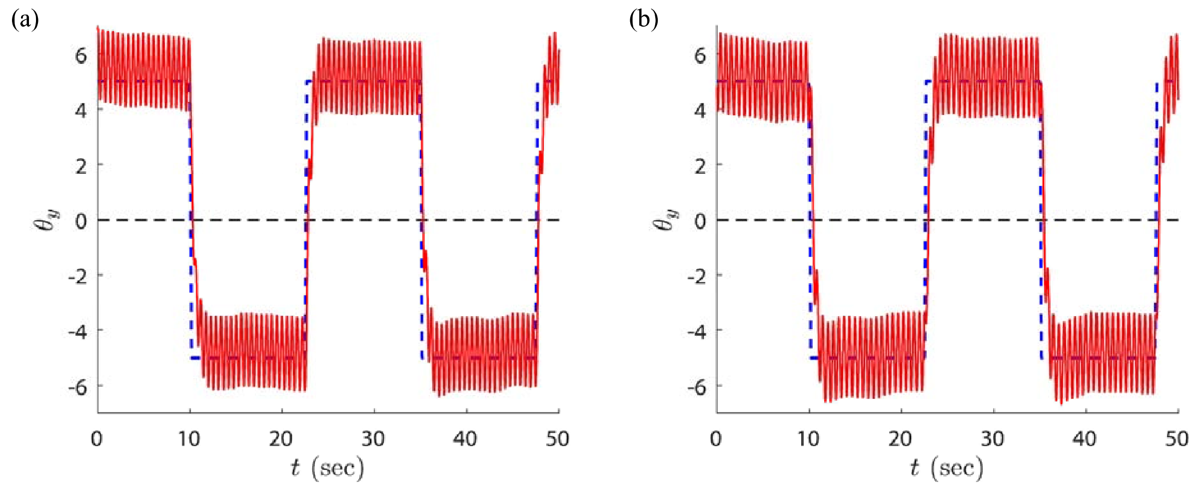

Fig. 7 Yaw angle ($\theta_y(t)$) of 3D hovercraft apparatus with feedback gains obtained using the proposed optimization-based strategy and delays of (a) 150 ms and (b) 160 ms. The reference signal is also shown (dashed line).

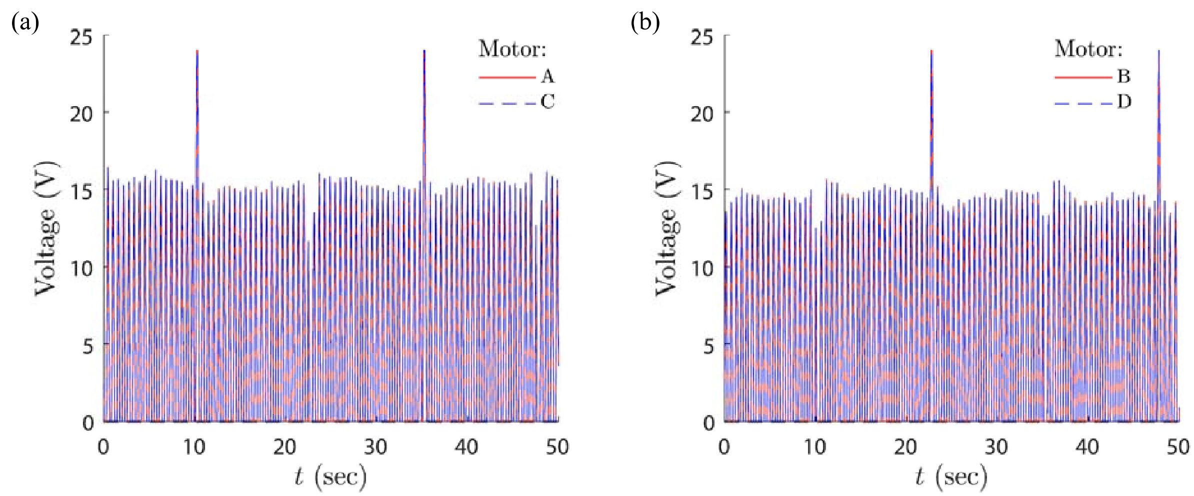

Fig. 8 Voltage of (a) motors A and C, and (b) motors B and D in the 3D hovercraft apparatus with feedback gains obtained using the proposed optimization-based strategy and a delay of 160 ms.

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