Spurious roots of delay differential equations using Galerkin approximations

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Abstract

The dynamics of time-delay systems are governed by delay differential equations (DDEs), which are infinite dimensional and can pose computational challenges. Several methods have been proposed for studying the stability characteristics of DDEs. One such method employs Galerkin approximations to convert DDEs into partial differential equations (PDEs) with boundary conditions; the PDEs are then converted into systems of ordinary differential equations (ODEs), whereupon standard ODE methods can be applied. The Galerkin approximation method can be applied to a second-order DDE in two ways: either by converting into a second-order PDE and then into a system of second-order ODEs (the "second-order Galerkin" method), or by first expressing as two first-order DDEs and converting into a system of first-order PDEs and then into a first-order ODE system (the "first-order Galerkin" method). In this paper, we demonstrate that these subtly different formulation procedures lead to different roots of the characteristic polynomial. In particular, the second-order Galerkin method produces spurious roots near the origin, which must then be identified through substitution into the characteristic polynomial of the original DDE. However, spurious roots do not arise if the first-order Galerkin method is used, which can reduce computation time and simplify stability analyses. We describe these two formulation strategies and present numerical examples to highlight their important differences.

Keywords

Delay differential equation, Galerkin approximation, Spurious root, Time-delay system

1 Introduction

Many dynamic processes can be described as time-delay systems (TDS), such as any process in which information or material propagates in finite time. The dynamics of such systems are governed by delay differential equations (DDEs): the state derivatives at the present time are explicit functions of past states. DDEs are infinite-dimensional systems and, as such, stability analysis and control design can be challenging for TDS.

To determine the stability of a DDE, one must determine the locations of its characteristic roots. Several methods exist for analyzing the stability of DDEs with constant delays, including the Lambert W function (Yi et al. (2010); Asl and Ulsoy (2003); Jarlebring and Damm (2007); Yi et al. (2007); Surya et al. (2018)), Galerkin approximations (Wahi and Chatterjee (2005); Vyasarayani (2012); Sadath and Vyasarayani (2015)), Laplace transforms (Kalmár-Nagy (2009)), semi-discretization (Insperger and Stépán (2011)), and pseudo-spectral collocation (Butcher et al. (2004); Breda et al. (2005); Wu and Michiels (2012)). The Galerkin approximation and pseudo-spectral collocation methods fall under the broad category of spectral methods. A key advantage of spectral methods is the convergence property of the approximate eigenvalues—specifically, the rightmost characteristic root of the DDE converges first (Vyasarayani et al. (2014)). The location of the rightmost characteristic root is critical when analyzing stability; thus, spectral methods are favorable in this context.

Galerkin approximations have been highly successful in studying the stability characteristics of second-order DDEs (Wahi and Chatterjee (2005); Vyasarayani (2012); Vyasarayani et al. (2014); Sadath and Vyasarayani (2015)).

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Galerkin approximations are applied to time-delay systems by first converting the governing DDE into a partial differential equation (PDE), then converting the PDE into a system of ordinary differential equations (ODEs). Sadath and Vyasarayani (2015) performed stability studies of second-order DDEs using Galerkin approximations, converting the second-order DDE into a second-order PDE and then into a system of second-order ODEs; only in the final step were the second-order ODEs rewritten as a system of first-order ODEs. We refer to this formulation procedure as the "second-order Galerkin" method. As observed by Sadath and Vyasarayani (2015), some eigenvalues of the approximating ODE system are *spurious*: they are not characteristic roots of the original DDE.

In some cases, spurious roots happen to be the rightmost eigenvalues of an approximating ODE system. Such roots pose analytical and practical challenges. For example, the rightmost root of a DDE is critical in determining its stability. If the rightmost root of an approximating ODE system might be spurious, then the approximating ODE system alone is insufficient to correctly determine the stability margin of the original DDE. When applying pole-placement techniques, additional analysis will be required for DDE systems whose approximating ODE systems contain spurious roots. It is also critical to correctly determine the rightmost spectrum when developing reduced-order models of DDEs; the presence of spurious roots complicates this process as well.

Spurious roots can be avoided using an alternative formulation, which begins by converting the second-order DDE into two first-order DDEs. The first-order DDEs are then converted into first-order PDEs and, finally, we arrive at a system of first-order ODEs (Chakraborty et al. (2019); Kandala et al. (2019)). We refer to this formulation procedure as the "first-order Galerkin" method. This formulation does not introduce spurious roots in the approximating ODE system because, as we will see, an artificial constraint is not imposed on the history functions. Note that stability charts obtained using the second-order and first-order Galerkin methods will be identical because the spurious roots introduced in the former are stable (i.e., they reside in the left half-plane). However, spurious roots must be identified and removed during pole placement and when calculating stability margin (the perpendicular distance between the rightmost root and the imaginary axis). This additional analysis can become computationally expensive for large systems. In short, an ideal formulation procedure will generate systems in which the rightmost root is authentic (i.e., not spurious), converges to the correct value, and is the first root to converge.

The remainder of this paper is organized as follows. In Section 2, we briefly outline the second-order and first-order Galerkin formulations. Numerical examples are presented in Section 3 to demonstrate their differences, and the work is summarized in Section 4.

2 Mathematical Modeling

Consider the following second-order DDE:

$$\ddot{x}(t) + a_1\dot{x}(t) + a_2x(t) + \sum_{r=1}^{n} b_r\dot{x}(t-\tau_r) + \sum_{q=1}^{m} c_q x(t-\tau_{n+q}) = 0, \quad (1)$$

where $x(t)$ is the system state vector, $\dot{x}(t)$ and $\ddot{x}(t)$ are its time derivatives, $a \in \mathbb{R}^2$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$, and delays $\tau_i \geq 0$ for $i = 1, 2, \ldots, n+m$. Equation (1) is a DDE if any $\tau_i > 0$; otherwise, it is simply an ODE. History functions, which describe the past states, are given as follows:

$$x(t) = \alpha(t), \quad -\bar{\tau} \leq t \leq 0, \quad (2a)$$

$$\dot{x}(t) = \beta(t), \quad -\bar{\tau} \leq t \leq 0, \quad (2b)$$

where $\bar{\tau} \triangleq \max(\tau)$. The Galerkin approximation method (Wahi and Chatterjee (2005); Vyasarayani (2012); Vyasarayani et al. (2014)) can be used to convert the infinite-dimensional DDE (Eq. (1)) into a finite-dimensional system of ODEs. The characteristic equation of the DDE is obtained by substituting $x(t) = x_0e^{\lambda t}$ into Eq. (1) and equating to zero:

$$D(\lambda) \triangleq \lambda^2 + a_1\lambda + a_2 + \sum_{r=1}^{n} b_r\lambda e^{-\lambda\tau_r} + \sum_{q=1}^{m} c_q e^{-\lambda\tau_{n+q}} = 0. \quad (3)$$

In the following, we outline the second-order and first-order Galerkin formulations.

2.1 Second-order Galerkin method

In this formulation, the governing second-order DDE is first converted into a second-order PDE; the PDE is then converted into a system of second-order ODEs which are finally written in first-order form. We briefly outline the formulation here (further detail can be found in Sadath and Vyasarayani (2015)). We begin by applying the transformation $y(s,t) = x(t+s)$. Differentiating this transformation with respect to t and, separately, with respect

to s reveals the following relation:

$$\frac{\partial y(s,t)}{\partial t} = \frac{\partial y(s,t)}{\partial s}, \quad -\bar{\tau} \leq s \leq 0, t \geq 0. \tag{4}$$

Differentiating Eq. (4) with respect to t results in a second-order PDE:

$$\frac{\partial^2 y(s,t)}{\partial t^2} = \frac{\partial^2 y(s,t)}{\partial t \partial s}, \quad -\bar{\tau} \leq s \leq 0, t \geq 0. \tag{5}$$

Thus, the initial value problem (Eqs. (1) and (2)) is re-cast as an initial–boundary value problem. Boundary conditions for Eq. (5) are obtained by substituting $s = 0$ and $s = -\bar{\tau}$ into the transformation $y(s,t) = x(t+s)$:

$$\begin{aligned} y(0,t) = x(t) \quad &\Rightarrow \quad \left.\frac{\partial y(s,t)}{\partial t}\right|_{s=0} = \dot{x}(t) \\ &\Rightarrow \quad \left.\frac{\partial^2 y(s,t)}{\partial t^2}\right|_{s=0} = \ddot{x}(t), \end{aligned} \tag{6a}$$

$$y(-\bar{\tau},t) = x(t-\bar{\tau}), \tag{6b}$$

and the initial conditions are obtained from the history functions (Eq. (2)):

$$y(s,0) = x(s) = \alpha(s), \quad -\bar{\tau} \leq s \leq 0, \tag{7a}$$

$$\dot{y}(s,0) = \dot{x}(s) = \beta(s), \quad -\bar{\tau} \leq s \leq 0. \tag{7b}$$

Notice that, by construction, history functions $\alpha(t)$ and $\beta(t)$ are now related through differentiation: from Eq. (7), we have $\beta(t) = \dot{\alpha}(t)$. This constraint was not present in the original problem statement and, as we will see below, is not present in the first-order Galerkin formulation. It is this artificial constraint that produces the spurious roots.

Briefly, the formulation concludes as follows. We assume a series solution and retain the first N terms:

$$\begin{aligned} y(s,t) &= \sum_{i=1}^{\infty} \phi_i(s)\eta_i(t) \\ &\approx \sum_{i=1}^{N} \phi_i(s)\eta_i(t) = \boldsymbol{\phi}^{\mathrm{T}}(s)\boldsymbol{\eta}(t), \end{aligned} \tag{8}$$

where $\boldsymbol{\phi}(s) \triangleq [\phi_1(s), \phi_2(s), \ldots, \phi_N(s)]^{\mathrm{T}}$ and $\boldsymbol{\eta}(t) \triangleq [\eta_1(t), \eta_2(t), \ldots, \eta_N(t)]^{\mathrm{T}}$ are the vectors of basis functions and coordinates, respectively. We substitute the truncated series solution into Eq. (5), pre-multiply by $\boldsymbol{\phi}(s)$, and integrate over the domain $s \in [-\bar{\tau}, 0]$ to obtain the following system of second-order ODEs:

$$\mathbf{M}\ddot{\boldsymbol{\eta}}(t) = \mathbf{C}\dot{\boldsymbol{\eta}}(t), \tag{9}$$

where $\mathbf{M} \triangleq \int_{-\bar{\tau}}^{0} \boldsymbol{\phi}(s)\boldsymbol{\phi}^{\mathrm{T}}(s)\,\mathrm{d}s$, $\mathbf{C} \triangleq \int_{-\bar{\tau}}^{0} \boldsymbol{\phi}(s)\boldsymbol{\phi}'(s)^{\mathrm{T}}\,\mathrm{d}s$, and $\boldsymbol{\phi}'(s)$ denotes the derivative of $\boldsymbol{\phi}(s)$ with respect to s. The boundary conditions for the original DDE (Eq. (1)) are given as follows:

$$\mathbf{m}\ddot{\boldsymbol{\eta}}(t) = \mathbf{c}\dot{\boldsymbol{\eta}}(t) + \mathbf{k}\boldsymbol{\eta}(t), \tag{10}$$

where $\mathbf{m}$, $\mathbf{c}$, and $\mathbf{k}$ are computed as follows:

$$\mathbf{m} = \boldsymbol{\phi}^{\mathrm{T}}(0), \tag{11}$$

$$\mathbf{c} = -a_1\boldsymbol{\phi}^{\mathrm{T}}(0) - \sum_{r=1}^{n} b_r\boldsymbol{\phi}^{\mathrm{T}}(-\tau_r), \tag{12}$$

$$\mathbf{k} = -a_2\boldsymbol{\phi}^{\mathrm{T}}(0) - \sum_{q=1}^{m} c_q\boldsymbol{\phi}^{\mathrm{T}}(-\tau_{n+q}). \tag{13}$$

The boundary conditions can be incorporated into Eq. (9) using the spectral-tau method (Vyasarayani et al. (2014)), ultimately resulting in a system of first-order ODEs:

$$\dot{\mathbf{r}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{K}} & \widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{C}} \end{bmatrix} \mathbf{r}(t) = \mathbf{G}_{\mathrm{so}}\mathbf{r}(t), \tag{14}$$

where $\mathbf{r}(t) \triangleq \left[\boldsymbol{\eta}^{\mathrm{T}}(t), \dot{\boldsymbol{\eta}}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{2N}$ is the state vector, and matrices $\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{C}}$ and $\widetilde{\mathbf{K}}$ are defined as follows:

$$\widetilde{\mathbf{M}} = \begin{bmatrix} \bar{\mathbf{M}} \\ \mathbf{m} \end{bmatrix}, \quad \widetilde{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{C}} \\ \mathbf{c} \end{bmatrix}, \quad \widetilde{\mathbf{K}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{k} \end{bmatrix}. \tag{15}$$

In Eq. (15), $\bar{\mathbf{M}}$ and $\bar{\mathbf{C}}$ are matrices of size $(N-1) \times N$ and are obtained by removing the last row from matrices $\mathbf{M}$ and $\mathbf{C}$, respectively. The solution of Eq. (14) approximates that of the original DDE (Eq. (1)), and the eigenvalues $(\hat{\lambda}_i, i = 1, 2, \ldots, 2N)$ of $\mathbf{G}_{\mathrm{so}}$ converge to the characteristic roots of Eq. (3) as N increases (Wahi and Chatterjee (2005); Vyasarayani et al. (2014)). We define error E_i as the absolute value of $D(\hat{\lambda}_i)$, obtained by substituting $\hat{\lambda}_i$ into Eq. (3). In this work, we consider root i to have converged if error $E_i < 10^{-6}$.

2.2 First-order Galerkin method

In this formulation, the governing second-order DDE is first converted into a system of first-order DDEs; we then obtain a system of first-order PDEs and, finally, arrive at a system of first-order ODEs (Chakraborty et al. (2019); Kandala et al. (2019)). We begin by defining state vector $\mathbf{w} \triangleq [x(t), \dot{x}(t)]^{\mathrm{T}}$

and rewriting Eq. (1) as follows:

$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) + \sum_{r=1}^{n} \mathbf{B}_r \mathbf{w}(t-\tau_r) + \sum_{q=1}^{m} \mathbf{Q}_q \mathbf{w}(t-\tau_{n+q}), \quad (16)$$

where matrices $\mathbf{A}$, $\mathbf{B}_r$, and $\mathbf{Q}_q$ are given as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad \mathbf{B}_r = \begin{bmatrix} 0 & 0 \\ 0 & -b_r \end{bmatrix}, \quad \mathbf{Q}_q = \begin{bmatrix} 0 & 0 \\ -c_q & 0 \end{bmatrix}. \quad (17)$$

We apply the transformation $\mathbf{y}(s,t) = \mathbf{w}(t+s)$ and differentiate with respect to t and, separately, with respect to s to obtain the following relation (analogous to Eq. (4)):

$$\frac{\partial \mathbf{y}(s,t)}{\partial t} = \frac{\partial \mathbf{y}(s,t)}{\partial s}, \quad -\bar{\tau} \le s \le 0, t \ge 0. \quad (18)$$

The boundary conditions for this first-order PDE (Eq. (18)) are obtained by substituting $s = 0$ and $s = -\bar{\tau}$ into the transformation $\mathbf{y}(s,t) = \mathbf{w}(t+s)$:

$$\mathbf{y}(0,t) = \mathbf{w}(t) \quad \Rightarrow \quad \left.\frac{\partial \mathbf{y}(s,t)}{\partial t}\right|_{s=0} = \dot{\mathbf{w}}(t), \quad (19a)$$

$$\mathbf{y}(-\bar{\tau},t) = \mathbf{w}(t-\bar{\tau}). \quad (19b)$$

Combining Eq. (19) with Eq. (16), we obtain the following:

$$\left.\frac{\partial \mathbf{y}(s,t)}{\partial t}\right|_{s=0} = \mathbf{A}\mathbf{y}(0,t) + \sum_{r=1}^{n} \mathbf{B}_r \mathbf{y}(-\tau_r,t) + \sum_{q=1}^{m} \mathbf{Q}_q \mathbf{y}(-\tau_{n+q},t). \quad (20)$$

Thus, we have converted the original first-order DDE (Eq. (1)) into an equivalent system of first-order PDEs (Eq. (18)) with boundary conditions given by Eq. (20).

We arrive at a system of first-order ODEs by assuming a series solution and retaining the first N terms:

$$y_i(s,t) = \sum_{j=1}^{\infty} \phi_j(s) z_{ij}(t) \approx \sum_{j=1}^{N} \phi_j(s) z_{ij}(t) = \boldsymbol{\phi}^{\mathrm{T}}(s)\mathbf{z}_i(t), \quad i = 1,2, \quad (21)$$

where $\boldsymbol{\phi}(s) \triangleq [\phi_1(s), \phi_2(s), \ldots, \phi_N(s)]^{\mathrm{T}}$ and $\mathbf{z}_i(t) \triangleq [z_{i1}(t), z_{i2}(t), \ldots, z_{iN}(t)]^{\mathrm{T}}$ are the vectors of basis functions and coordinates, respectively. We define $\boldsymbol{\Psi}(s) \in \mathbb{R}^{2N\times 2}$ and $\boldsymbol{\beta}(t) \in \mathbb{R}^{2N\times 1}$ as follows:

$$\boldsymbol{\Psi}(s) = \begin{bmatrix} \boldsymbol{\phi}(s) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi}(s) \end{bmatrix}, \quad \boldsymbol{\beta}(t) = \begin{Bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{Bmatrix}. \quad (22)$$

Equation (21) can then be rewritten as $\mathbf{y}(s,t) = \boldsymbol{\Psi}^{\mathrm{T}}(s)\boldsymbol{\beta}(t)$ and substituted into Eq. (18) to arrive at the following:

$$\boldsymbol{\Psi}^{\mathrm{T}}(s)\dot{\boldsymbol{\beta}}(t) = \boldsymbol{\Psi}'(s)^{\mathrm{T}}\boldsymbol{\beta}(t). \quad (23)$$

Pre-multiplying Eq. (23) by $\boldsymbol{\Psi}(s)$ and then integrating over the domain $s \in [-\bar{\tau}, 0]$ produces a system of first-order ODEs:

$$\mathbf{P}\dot{\boldsymbol{\beta}}(t) = \mathbf{R}\boldsymbol{\beta}(t). \quad (24)$$

Matrices $\mathbf{P}$ and $\mathbf{R}$ are square, block-diagonal, and of dimension $2N$:

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{(2)} \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{(2)} \end{bmatrix}^{\mathrm{T}}, \quad (25)$$

where submatrices $\mathbf{P}^{(i)}$ and $\mathbf{R}^{(i)}$ are defined as follows:

$$\mathbf{P}^{(i)} \triangleq \int_{-\bar{\tau}}^{0} \boldsymbol{\phi}_i(s)\boldsymbol{\phi}_i^{\mathrm{T}}(s)\,\mathrm{d}s, \quad \mathbf{R}^{(i)} \triangleq \int_{-\bar{\tau}}^{0} \boldsymbol{\phi}_i(s)\boldsymbol{\phi}_i'(s)^{\mathrm{T}}\,\mathrm{d}s, \quad i = 1,2. \quad (26)$$

The matrix of boundary conditions is obtained by substituting the series solution (Eq. (21)) into Eq. (20):

$$\boldsymbol{\Psi}^{\mathrm{T}}(0)\dot{\boldsymbol{\beta}}(t) = \left[\mathbf{A}\boldsymbol{\Psi}^{\mathrm{T}}(0) + \sum_{r=1}^{n} \mathbf{B}_r \boldsymbol{\Psi}^{\mathrm{T}}(-\tau_r) + \sum_{q=1}^{m} \mathbf{Q}_q \boldsymbol{\Psi}^{\mathrm{T}}(-\tau_{n+q})\right]\boldsymbol{\beta}(t). \quad (27)$$

Equations (24) and (27) can be combined as follows:

$$\mathbf{U}\dot{\boldsymbol{\beta}}(t) = \mathbf{V}\boldsymbol{\beta}(t) \quad (28a)$$

$$\Rightarrow \quad \dot{\boldsymbol{\beta}}(t) = \mathbf{U}^{-1}\mathbf{V}\boldsymbol{\beta}(t) \triangleq \mathbf{G}_{\mathrm{fo}}\boldsymbol{\beta}(t), \quad (28b)$$

where $\mathbf{U}$ and $\mathbf{V}$ are square matrices of dimension $2N$ and are obtained by replacing the Nth and $2N$th rows of Eq. (24) with the first and second rows of Eq. (27), respectively. The solution of Eq. (28) approximates that of the original DDE (Eq. (1)), and the eigenvalues ($\hat{\lambda}_i, i = 1,2,\ldots,2N$) of $\mathbf{G}_{\mathrm{fo}}$ converge to the characteristic roots of Eq. (3) as N increases (Wahi and Chatterjee (2005); Vyasarayani et al. (2014)). We define error E_i and the criterion for determining convergence as described above.

In spectral methods, one must select an appropriate set of basis functions $\boldsymbol{\phi}(s)$ for obtaining the solution of

the approximating ODE system. We use shifted Legendre polynomials, as they have been shown to have good convergence properties (Vyasarayani et al. (2014)):

$$\phi_1(s) = 1, \tag{29a}$$

$$\phi_2(s) = 1 + \tfrac{2s}{\bar{\tau}}, \tag{29b}$$

$$\phi_k(s) = \frac{(2k-3)\,\phi_2(s)\phi_{k-1}(s) - (k-2)\,\phi_{k-2}(s)}{k-1}, \quad k = 3, 4, \ldots, N. \tag{29c}$$

With these basis functions, matrices $\mathbf{M}$ and $\mathbf{C}$ in the second-order formulation (Eq. (9)) and matrices $\mathbf{P}$ and $\mathbf{R}$ in the first-order formulation (Eq. (25)) can be expressed in closed form, as reported previously by Sadath and Vyasarayani (2015). For both second-order and first-order formulations, considering N terms in the series solution (Eqs. (8) and (21)) results in approximately $N/2$ converged eigenvalues.

3 Results and Discussion

In this section, we present two examples that compare the results obtained using the second-order and first-order Galerkin formulations.

3.1 Example 1

Consider the following second-order DDE with delay $\tau = 1$:

$$\ddot{x}(t) + \dot{x}(t) + x(t) + x(t-\tau) = 0. \tag{30}$$

We use $N = 50$ terms in the series solution (i.e., $\mathbf{G}_{\mathrm{so}}$ and $\mathbf{G}_{\mathrm{fo}}$ are each of dimension $2N = 100$). Figure 1 shows the rightmost eigenvalues of $\mathbf{G}_{\mathrm{so}}$ (Eq. (14)) and $\mathbf{G}_{\mathrm{fo}}$ (Eq. (28)). In each case, we obtain 22 converged roots—that is, $\mathbf{G}_{\mathrm{so}}$ and $\mathbf{G}_{\mathrm{fo}}$ each have 22 eigenvalues $\hat{\lambda}_i$. When substituted into the characteristic equation of the original DDE (Eq. (3)), we obtain an error of $E_i < 10^{-6}$. However, as shown in Fig. 1(a), the second-order formulation produces spurious roots—in fact, there are 49 spurious roots very close to the origin. Note that, in this example, the spurious roots happen to be the rightmost eigenvalues of $\mathbf{G}_{\mathrm{so}}$ and could be misinterpreted as the rightmost eigenvalues of the original DDE. Since the second-order formulation is known to produce spurious roots, each root must therefore be substituted into the characteristic equation (Eq. (3)) to ensure its authenticity. The absence of spurious roots in the first-order formulation guarantees that the roots converge starting with the rightmost characteristic root of the DDE, which avoids the additional step of computing E_i (Chakraborty et al. (2019)) and makes this formulation particularly useful for solving pole-placement problems (Kandala et al. (2019)).

3.2 Example 2

Consider the following second-order DDE with delays $\tau_1 > 0$ and $\tau_2 > 0$:

$$\ddot{x}(t) + \dot{x}(t) + x(t) + \dot{x}(t-\tau_1) + x(t-\tau_2) = 0. \tag{31}$$

We again use $N = 50$ terms in the series solution. Figure 2 shows the rightmost eigenvalues of $\mathbf{G}_{\mathrm{so}}$ and $\mathbf{G}_{\mathrm{fo}}$ for $\tau_1 = \tau_2 = 1$. In this example, we obtain 23 converged roots using each formulation. As shown in Fig. 2(a), the second-order formulation again produces spurious roots: there are 49 spurious roots very close to the origin, and these are once again the rightmost roots of $\mathbf{G}_{\mathrm{so}}$. The spurious roots are stable (i.e., they reside in the left half-plane), so they do not affect the stability chart (Fig. 3). In fact, as shown in Fig. 4 for $\tau_1 = \tau_2 = 3$, the spurious roots are stable even if the closed-loop system is unstable. Nevertheless, as mentioned previously, it is generally favorable to avoid spurious roots.

3.3 Source of spurious roots

We revisit Eq. (14):

$$\dot{\mathbf{r}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{K}} & \widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{C}} \end{bmatrix} \mathbf{r}(t) = \mathbf{G}_{\mathrm{so}}\mathbf{r}(t), \tag{32}$$

where matrices $\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{C}}$ and $\widetilde{\mathbf{K}}$ are defined in Section 2.1. Close inspection of $\widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{K}}$ reveals that its entries are as follows:

$$\left[\widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{K}}\right]_{ij} \begin{cases} = 0, & \text{if } 1 \le i \le N-1 \text{ and } 1 \le j \le N, \\ = 0, & \text{if } i = N \text{ and } j \text{ is even}, \\ \neq 0, & \text{if } i = N \text{ and } j \text{ is odd}, \end{cases} \tag{33}$$

where $i \in [1, N]$ and $j \in [1, N]$ are the row and column indices, respectively. It should be noted that, in matrix $\mathbf{G}_{\mathrm{so}}$, the entries in rows 1 to $2N-1$ and columns 1 to N are zeros. This means that $\mathbf{G}_{\mathrm{so}}$ has N linearly dependent columns and the system has $N-1$ eigenvalues at zero. However, matrix $\mathbf{G}_{\mathrm{fo}}$ has full rank (i.e., $2N$ linearly independent columns) and, consequently, none of the system's eigenvalues are zero. These observations can be confirmed using Example 2. Considering $N = 4$ and $\tau_1 = \tau_2 = 1$, we compute matrices

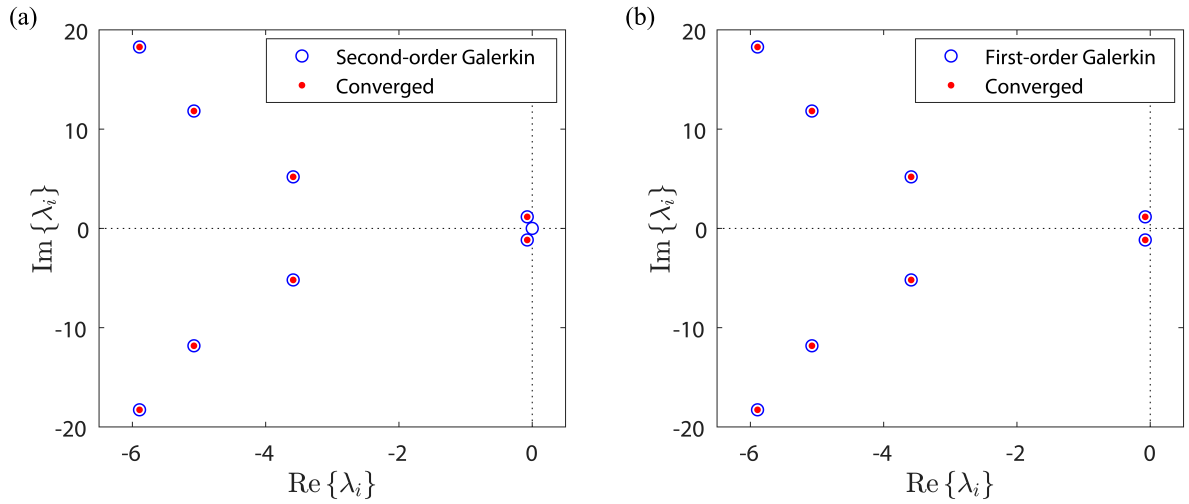

Figure 1. Characteristic roots of Eq. (30) obtained using the second-order (a) and first-order (b) Galerkin formulations.

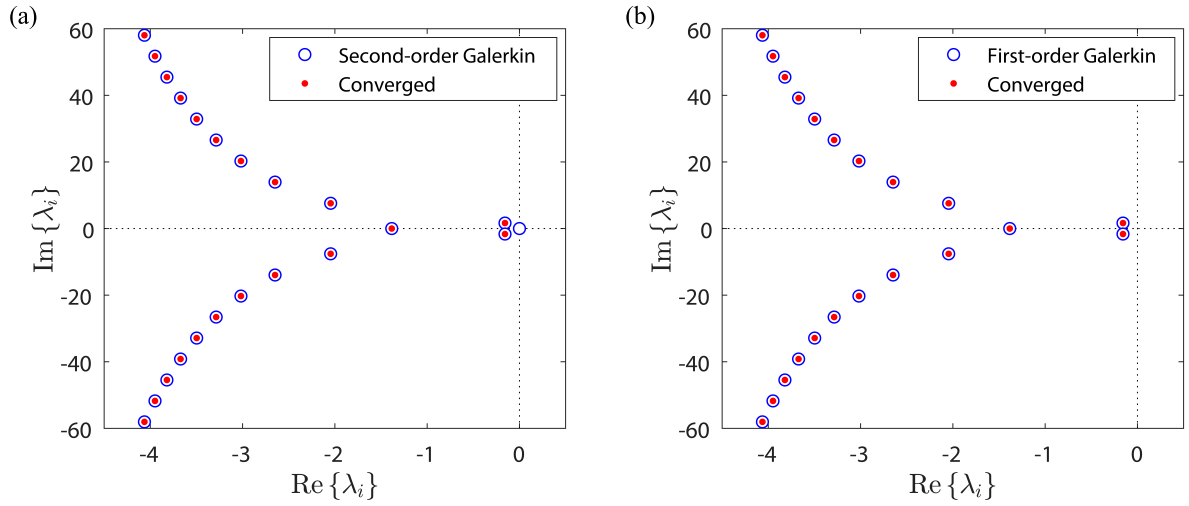

Figure 2. Characteristic roots of Eq. (31) with $\tau_1 = \tau_2 = 1$ obtained using the second-order (a) and first-order (b) Galerkin formulations.

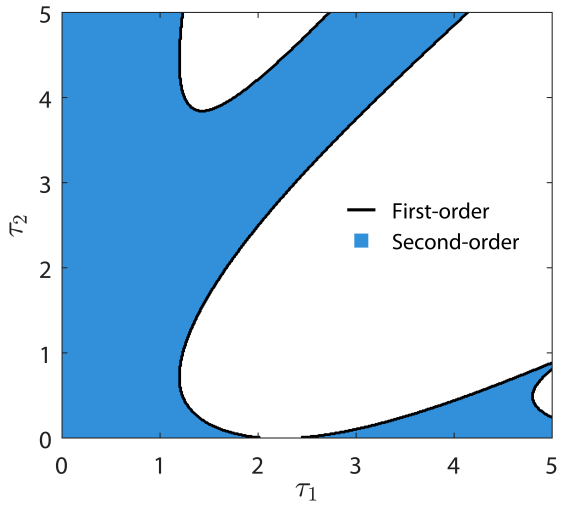

Figure 3. Stability chart of Eq. (31) with $\tau_1 = \tau_2 = 1$ obtained using the first-order (black line) and second-order (shaded region) Galerkin formulations.

$\mathbf{G}_{\mathrm{so}}$ and $\mathbf{G}_{\mathrm{fo}}$ as follows:

$$\mathbf{G}_{\mathrm{so}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ -2 & 0 & -2 & 0 & -2 & -2 & -8 & -12 \end{bmatrix} \tag{34a}$$

$$\mathbf{G}_{\mathrm{fo}} = \begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & -2 & -6 & -12 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ -2 & 0 & -2 & 0 & -2 & -2 & -8 & -12 \end{bmatrix} \tag{34b}$$

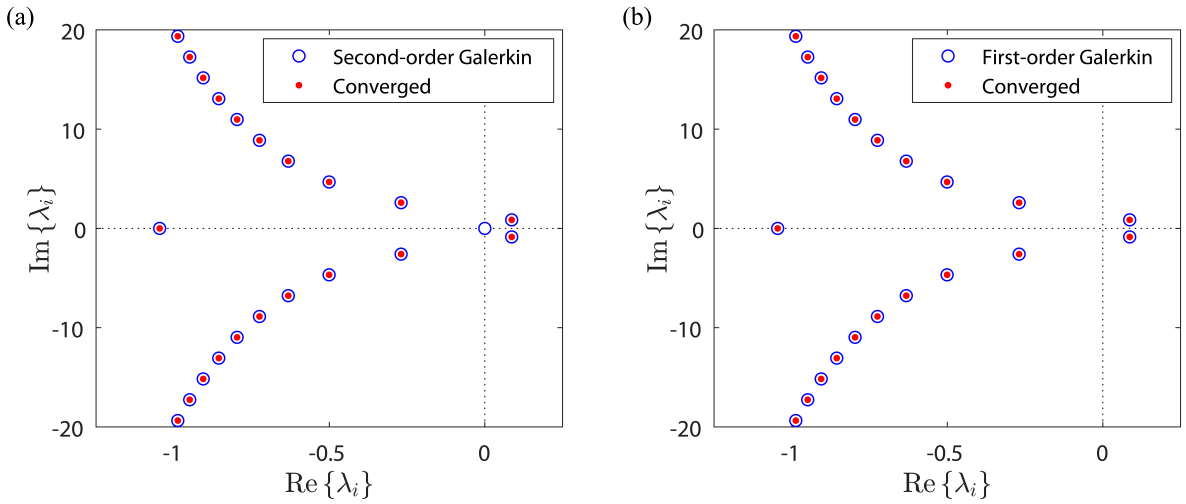

Figure 4. Characteristic roots of Eq. (31) with $\tau_1 = \tau_2 = 3$ obtained using the second-order (a) and first-order (b) Galerkin formulations.

The eigenvalues of $\mathbf{G}_{\text{so}}$ that are zeros appear as $N-1$ spurious roots at the origin, as can be seen in Figs. 1(a), 2(a), and 4(a).

4 Conclusions

In this work, we have compared two approaches for applying the Galerkin approximation method to second-order DDEs. The "second-order Galerkin" formulation involves converting the DDE into a second-order PDE, then into a system of second-order ODEs, and finally rewriting the ODEs in first-order form. The "first-order Galerkin" formulation involves converting the second-order DDE into two first-order DDEs, then into a system of first-order PDEs, and finally into a system of first-order ODEs. The limitations of the second-order formulation have been demonstrated both analytically and numerically with two examples. In particular, the second-order formulation imposes an artificial constraint between the two history functions, which ultimately precipitates into spurious roots in the approximating ODE system. The spurious roots are stable and, thus, do not affect stability charts; however, in reduced-order modeling and pole-placement problems, additional analysis must be performed to determine whether a root is spurious. This issue has not been reported previously, and one should be mindful of it when using this technique. The rank-deficient matrix that arises in the second-order formulation can be avoided by using the first-order formulation. As we demonstrate, the first-order formulation produces the same stability chart but no spurious roots. A substantial benefit is that the rightmost roots are true characteristic roots, which are the first to converge. We are currently investigating the stability of linear DDEs using Floquet theory and, in particular, the presence of spurious Floquet multipliers.

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